Quantum properties of a toroidal compactified spacetime outside a cosmic string

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We investigate some quantum properties of a toroidal compactified spacetime modified due this unusual topology. More specifically we present the vacuum fluctuation and the particle production in a Friedmann cosmological model in $\mathbb{R}^3 \times S^1$ outside a $U(1)$ cosmic string, which throughout this paper, means the local type of these topological defects. The case of a teleparallel Friedmann spacetime is investigated, where we analyze the case with torsion. Also, we present a generalization to toroidal compactification of p extra dimensions, where the topology is given by $\mathbb{R}^3 \times T^p$. Some implications are presented and discussed. Thoroughout the paper we are not solely interested in the dynamics of spacetime, but in the physical consequences of the topological transformations.

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Introduction

Cosmic strings are a very interesting class of topological defects, which arise from spontaneous symmetry breaking. Depending on the symmetry that is broken, the kind of string can change [1]. At the same time, up to now there is no consistent physical law that determines the topological shape of universe. Here we investigate the fluctuation of stress energy tensor and the particle production due to an expansion of the universe given by the following metric ($G = c = \hbar = 1$)

$$ds^2 = a(t)(-dt^2 + dr^2 + B^2 r^2 d\theta^2 + dz^2), \qquad (1)$$

where the z direction is compactified in a circle — S^1 topology — and the string is chosen to be along this direction has a finite length, say L [2]. In Eq.(1) the parameter B is related to the string linear density μ by $B = 1 - 4\mu$ and in this paper we consider the GUT scale ($\mu \sim 10^{-6}$). With these assumptions our coordinate system obeys the equivalence class constraint $(t, r, \theta, z) = (t, r, \theta, z + mL)$, where m is an integer. In this case, Eq.(1) represents the line element outside a straight cosmic string in $r = 0$, parallel to the z-axis in a Friedmann cosmological model with $S^1 \times \mathbb{R}^3$ topology.

In what follows we calculate the components of stress tensor for a scalar massless field in the minimal coupling, analyze the particle production, including the teleparallel case, in the context of Eq.(1) and then generalize these results to the $\mathbb{R}^3 \times T^p$ topology. The article is presented as follows: in Section (I) we calculate the vacuum fluctuation via the stress tensor for a massless scalar field, showing from the explicit form of the expected value of the stress tensor that Lorentz invariance is not preserved at the quantum level. In Section (II) we analyze the particle production using the solutions of the Klein-Gordon equation to calculate the Bogoliubov coefficients. In Section (III) the torsion case is incorporated via a teleparallel connection and finally in Section (IV) the previous cases are generalized for a p-dimensional toroidal compactification.

I. VACUUM FLUCTUATION

In order to make a complete analysis of quantum vacuum outside (1) in the *in* and *out* states ($a(t)$ cte.), we need to compute the fluctuation of the stress tensor $\langle \hat{T}^\nu_\mu(x)\rangle$. The origin of non-vanishing components of $\langle \hat{T}^\nu_\mu\rangle$ outside a cosmic string is well known to be purely topological [3, 4]. It is then expected that in our topology other unusual effects of topological origin can be unravelled.

The stress tensor for a scalar massless field in the minimally coupled case is given by [5]

$$\langle \hat{T}^\nu_\mu\rangle = \lim_{x'\to x}(\nabla_\mu \nabla^{\nu'} - \frac{1}{2}\nabla_\mu \nabla^\nu)G(x, x'), \qquad (2)$$

where ∇_μ and $\nabla_{\mu'}$ means respectively differentiation with respect to x and x' coordinates, and $G(x, x')$ denotes the propagator. In our case is possible to show

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that the two point Green's function is given by [3, 8]

$$G(x,x') = \frac{1}{8\pi^3 B}\int_0^\infty \frac{F_B(u,\theta-\theta')}{R^2-(t-t')^2}du, \tag{3}$$

where $R = [r^2 + r'^2 + (z-z')^2 + 2rr'\cosh u]^{1/2}$ and

$$\begin{aligned} F_B(u,\psi) &= \frac{\sin\psi - \pi/B}{\cosh u/B - \cos\psi - \pi/B} \\ &\quad - \frac{\sin\psi + \pi/B}{\cosh u/B - \cos\psi + \pi/B}. \end{aligned}$$

However the periodicity along z direction (S^1 topology) enables us to impose $x' \mapsto x'_m = (t', r', \theta', z' + mL)$. Then the Green's function needs to be modified by

$$G(x,x') \mapsto \sum_{m=-\infty}^{+\infty} G(x, x'_m), \tag{4}$$

with $m \neq 0$.

We emphasize that the propagator is regular and, moreover, valid only in the range $-\pi/B < \theta - \theta' < \pi/B$, and at least formally it is possible to calculate the components of the stress tensor. However, it leads to complicated integrals. Below the expressions of non-vanishing components are given. In the diagonal we have

$$\begin{aligned} \langle \hat{T}(x)^0_0 \rangle &= \frac{\sin\pi/B}{8\pi^3 B}\sum_{m=-\infty}^{+\infty}\int_0^\infty \frac{du}{\Im_{B,L}(u)} \\ &\times \left(\frac{1}{D(u)}\big(8r^2 - 8m^2L^2\right. \\ &+ 2\cosh u[2r^2(3+\cosh u) - m^2L^2]\big) \\ &+ \left.\Re_B(u)\right), \end{aligned} \tag{5}$$

$$\begin{aligned} \langle \hat{T}(x)^1_1 \rangle &= \frac{\sin\pi/B}{8\pi^3 B}\sum_{m=-\infty}^{+\infty}\int_0^\infty \frac{du}{\Im_{B,L}(u)} \\ &\times \left(\frac{1}{D(u)}\big(-4m^2L^2\right. \\ &- 2\cosh u[2r^2(1+\cosh u) - m^2L^2]\big) \\ &+ \left.\Re_B(u)\right), \end{aligned} \tag{6}$$

$$\begin{aligned} \langle \hat{T}(x)^2_2 \rangle &= \frac{\sin\pi/B}{8\pi^3 B}\sum_{m=-\infty}^{+\infty}\int_0^\infty \frac{du}{\Im_{B,L}(u)} \\ &\times \left(\frac{1}{D(u)}\big(16r^2 - 4m^2L^2\right. \\ &+ 2\cosh u[2r^2(3+\cosh u) - m^2L^2]\big) \\ &+ \left.\Re_B(u)\right) \end{aligned} \tag{7}$$

and

$$\begin{aligned} \langle \hat{T}(x)^3_3 \rangle &= \frac{\sin\pi/B}{8\pi^3 B}\sum_{m=-\infty}^{+\infty}\int_0^\infty \frac{du}{\Im_{B,L}(u)} \\ &\times \left(\frac{1}{D(u)}\big(8r^2 + 8m^2L^2\right. \\ &+ 2\cosh u[2r^2(3+\cosh u) - m^2L^2]\big) \\ &+ \left.\Re_B(u)\right), \end{aligned} \tag{8}$$

where the functions $\Im_{B,L}(u)$, $\Re_B(u)$ and $D(u)$ are, respectively given by

$$\begin{aligned} \Im_{B,L}(u) &= (\cosh u/B - \cos\pi/B) \\ &\times [2r^2(1+\cosh u) + m^2L^2], \end{aligned}$$

$$\Re_B(u) = \frac{-2 + \cosh u/B(\cosh u/B - \cos\pi/B)}{(\cosh u/B - \cos\pi/B)^2}$$

and

$$D(u) = [2r^2(1+\cosh u) + m^2L^2]^2.$$

Note that the Eqs.(5) and (8) do not equal each other, meaning the breaking of Lorentz boosts invariance in the z direction at quantum level. It is indeed an explicit feature of nontrivial topology, since without compactification the invariance is still valid. Apart of this, there is another effect of $S^1 \times \mathbb{R}^3$ topology. Out of diagonal there are two non-vanishing components given by

$$\begin{aligned} \langle \hat{T}(x)^3_1 \rangle = \langle \hat{T}(x)^1_3 \rangle &= \frac{-2Lr\sin\pi/B}{\pi^3 B}\sum_{m=-\infty}^{+\infty} \\ &\times \int_0^\infty \frac{m(1+\cosh u)du}{\Im_{B,L}(u)}. \end{aligned}$$

Note that in the limit $L \to 0$, there are no contributions from $\langle \hat{T}(x)^3_1 \rangle$. The next step is to compute the particle production due to the expansion of universe in this cosmological model.

II. PARTICLE PRODUCTION

When the metric is static, even in curved spaces it is easy to define the vacuum state of the system. This is because it is trivial to find a time-like Killing vector that generates an one-parameter Lie group of isometries and so the vacuum and all Fock space can be defined. Let us analyze the particle production of a scalar massless field in our model due an expansion of our universe, i. e., corresponding a functional form to $a(t)$ in Eq.(1).

Our *in* and *out* states are defined by the static background (cosmic string) and the expansion will be implemented by a special function in the conformal factor.

As expected, the nontrivial topology implies a change in physical events. The procedure is very similar to the one in [6, 7].

The Klein-Gordon equation for the massless scalar field is given by

$$\left(\partial_t^2 + \frac{1}{a}\partial_t a\partial_t - \frac{1}{r}\partial_r(r\partial_r) - \frac{1}{B^2r^2}\partial_\theta^2 - \partial_z^2\right)\phi(t,r,\theta,z) = 0. \tag{9}$$

After separation of variables we have that $\phi = R(r)e^{i\lambda z}e^{i\alpha\theta}T(t)$, where $T(t)$ and $R(r)$ respectively satisfies the following equations:

$$\frac{d^2T(t)}{dt^2} + \frac{\dot{a}}{a}\frac{dT(t)}{dt} + w^2T(t) = 0 \tag{10}$$

and

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} - \frac{\alpha^2R(r)}{B^2r^2} + (\lambda^2 - w^2)R(r) = 0\ , \tag{11}$$

where $\dot{a} = da/dt$ and w is a separation constant. Note that α and λ obeys periodical constraints due to the topology of the cosmic string and S^1, respectively. Dirichlet boundary conditions are imposed at $r = \tilde{r}$ to keep the produced energy in a limited region [9]. Then the solution of Eq.(11) is given by

$$R(r) = \frac{J_\nu(\sqrt{\lambda^2 - w^2}r)}{J_\nu(\sqrt{\lambda^2 - w^2}\tilde{r})} - \frac{Y_\nu(\sqrt{\lambda^2 - w^2}r)}{Y_\nu(\sqrt{\lambda^2 - w^2}\tilde{r})}, \tag{12}$$

where $\nu = \alpha/B$. Besides, the solution on the string is well-defined by imposing an additional vanishing boundary condition at $r = r_0$, our solution is valid in the range $r_0 < r < \tilde{r}$. In this way the values of w can be found by the transcendental equation that arises from the boundary conditions ($C = \sqrt{\lambda^2 - w^2}$)

$$J_\nu(Cr_0)Y_\nu(C\tilde{r}) - J_\nu(C\tilde{r})Y_\nu(Cr_0) = 0 \tag{13}$$

and the roots are labeled by $k \in \mathbb{N}$. To find the Bogoliubov coefficients we need to know the function $T(t)$ in the *in* and *out* regions. It is briefly recalled here, and for more details see [5]. After a change in time coordinate by $\tau = \int \frac{dt}{a(t)}$ we have

$$\frac{d^2T(\tau)}{d\tau^2} + w^2a^2(\tau)T(\tau) = 0. \tag{14}$$

Now we suppose a smooth expansion for this universe, e.g., $a(\tau) = \left(\frac{1}{2}[\Omega^2 + 1 + (\Omega^2 - 1)\tanh(\rho\tau)]\right)^{1/2}$ where Ω is a constant and ρ gives the rate of expansion. Finally the solution is given in terms of hyperbolic and hypergeometric functions

$$\begin{aligned} T_{\frac{in}{out}}(\tau) &= \frac{1}{4\pi^2}(4\pi w_{\frac{in}{out}})^{-1/2} \\ &\times \exp\left(-iw_+\tau - (iw_-/\rho)\ln[2\cosh\rho\tau]\right) \\ &\times\ {}_2F_1\Big(1 + (iw_-/\rho), iw_-/\rho; 1 \\ &\mp (iw_{\frac{in}{out}}/\rho); \frac{1}{2}(1 \pm \tanh\rho\tau)\Big) \end{aligned}$$

where $w_{in} = w$, $w_{out} = w\Omega$ and $w_\pm = \frac{w}{2}(\Omega \pm 1)$. The linear transformations of hyperbolic functions give us the desired relation between *in* and *out* states

$$\phi^{in}_{\lambda,\alpha,k} = \gamma(\lambda,\alpha,k)\phi^{out}_{\lambda,\alpha,k} + \beta(\lambda,\alpha,k)(\phi^{out}_{\lambda,\alpha,k})^*.$$

The terms γ and β above are the so-called Bogoliubov coefficients given, in our case, by

$$\gamma(\lambda,\alpha,k) = \Omega\frac{\Gamma[1 - iw_{in}/\rho]\Gamma(-iw_{out}/\rho)}{\Gamma(-iw_+/\rho)\Gamma[1 - iw_+/\rho]} \tag{15}$$

and

$$\beta(\lambda,\alpha,k) = \Omega\frac{\Gamma[1 - iw_{in}/\rho]\Gamma(iw_{out}/\rho)}{\Gamma(iw_-/\rho)\Gamma[1 + iw_-/\rho]}. \tag{16}$$

According to usual theory in curved spaces, the density of created particles per mode is $|\beta(\lambda,\alpha,k)|^2$, i. e.,

$$|\beta(\lambda,\alpha,k)|^2 = \frac{\sinh^2(\pi/\rho w_-)}{\sinh(\pi/\rho w_{in})\sinh(\pi/\rho w_{out})}. \tag{17}$$

We should remark that the effects of nontrivial topology is reflected on the excitation modes. In Eq.(17) the modes assigned by α must obey the periodic constraint

$$\alpha = 2\pi n. \tag{18}$$

Besides, the S^1 compactification in the z coordinate implements another constraint, this time in λ, since the string have a finite length L and these modes have a discrete spectrum

$$\lambda = \frac{2\pi m}{L} \tag{19}$$

where $n, m \in \mathbb{Z}$.

III. THE TORSION CASE

The Klein-Gordon equation for the massless scalar field in a Friedmann teleparallel spacetime, where there exists torsion — represented by the torsion tensor $T^{\nu}{}_{\rho}{}^{\lambda}$ — is given by

$$\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi) + T^{\nu}{}_{\nu}{}^{\lambda}\partial_\lambda\phi = 0, \tag{20}$$

After separation of variables, only the radial and temporal functions are modified by torsion effects in the teleparallel case, and we get

$$\phi(t, r, \theta, z) = T(t)R(r)\theta(\vartheta)Z(z), \tag{21}$$

where

$$R(r) = \frac{1}{\sqrt{r}}\left(C_1\, J_{\frac{1}{2}\sqrt{1-4n^2}}(-ikr) + C_2\, Y_{\frac{1}{2}\sqrt{1-4n^2}}(-ikr)\right), \tag{22}$$

and in the particular case where $n = 0$, $R(r)$ reads

$$R(r) = A_1\, \exp(kr) + A_2\, \exp(-kr). \tag{23}$$

We have to choose in Eq.(22) the term which is regular at the origin.

The temporal equation is given by

$$\frac{\partial^2 T(t)}{\partial t^2} + 2\dot{a}(t)\sqrt{a(t)} + \frac{\partial T(t)}{\partial t} + a(t)\omega^2 T(t) = 0. \tag{24}$$

Note that there is an extra term arising from torsion effects. There is not in general an analytical function which is a solution of the equation above, but in some particular cases of $a(t)$ the solution is analytical. For instance, the solution when $a(t) = \tanh(\ln t)$ is given by

$$T(t) = A\, \cos(f(t)^{1/4} \exp[2f(t)]t + \delta) \tag{25}$$

where δ is a phase and $f(t) = \sqrt{\frac{bt^2-1}{bt^2+1}}$ (b=cte.). Also,

$$\theta(\vartheta) = D_1\, \cos(k\vartheta) + D_2\, \sin(k\vartheta) \tag{26}$$

and

$$Z(z) = \exp(i\alpha z), \tag{27}$$

where the condition $Z(z) = Z(z + 2m\pi)$ — in order to accomplish the toroidal compactification — is implicit in the form of Eq.(27).

IV. EXTRA DIMENSIONS

Once we have introduced the formalism for one compactified dimension,

the generalization to extra dimensions is immediate. First of all, we introduce a topological structure with cylindrical symmetry in the hypersurface (t, z_i) constant. Our interest resides on the kinematics of spacetime which topological transformations. We start with the following line element

$$\begin{aligned} ds^2 &= a(t)(-dt^2 + dr^2 + B^2 r^2 d\theta^2 \\ &+ b(t)/a(t) dz_i^2)\ , \end{aligned} \tag{28}$$

where $i = 1, \ldots, p$. Now the topology is $\mathbb{R}^3 \times T^p$ so the extra dimensions are also compactified (just like z) and we can investigate the particle production for another formalism. Note that we are not considering a cloud of strings in the sense of [10]. Instead, we just deal with a unusual topology without any string in extra dimensions. Suppose that our universe, after the expansion analyzed before, passes for another expansion behavior characterized just by the dimensions of the torus. In other words let us say that in a first moment $b(t) = a(t)$, in such way that all previous analysis is still valid. After it, in $t = t_0$ consider an expansion in the extra dimensions and in the z direction. The Klein-Gordon equation for Eq.(28) gives

$$\begin{aligned} &\left[\partial_t^2 + \frac{1}{2}\Big(\frac{\dot{b}}{b} + \frac{\dot{a}}{b}\Big)\partial_t\right]\phi + \frac{1}{r}[\partial_r(r\partial_r)]\phi \\ &+ \ \frac{1}{r^2 B^2}\partial_\theta^2\phi + \frac{a(t)}{b(t)}\partial_{z_i}^2\phi = 0 \end{aligned} \tag{29}$$

Note that if $b(t) = a(t)$ and $p = 1$ we recover the previous situation. If $b(t) = a(t)$ and $p \neq 1$ the generalization is immediate, where it is enough to make the product $\prod_i \exp(\lambda_i z_i)$ in the solution. Of course, there will be another classification for each root of Eq.(13).

Now consider the case when $a = \Omega$ and $b(t)$ simulate an expansion until $t = \bar{t}$. Then in the range $t_0 < t < \bar{t}$, after the expansion in all our universe ($a(t)$) we have

$$\frac{d^2 T}{dt^2} + \frac{1}{2}\frac{\dot{b}}{b}\frac{dT}{dt} + \left(\frac{\Omega}{b}\lambda_i^2 + w^2\right)T = 0 \tag{30}$$

and

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(w^2 - \frac{\alpha^2}{r^2 B^2}\right)R = 0. \tag{31}$$

The separation of variables is a little bit different from previous case and now t_0 and $\bar{t}$ will give our asymptotic solutions. The solution for Eq.(31) is given by

$$R(r) = C_1 J_{(\alpha/B)}(wr) + C_2 Y_{(\alpha/B)}(wr), \tag{32}$$

where C_1 and C_2 are constants to be determined while Eq.(30) gives, for a expansion implemented for $e^{b_0 t}$, the solution

$$T(t) = e^{\frac{-b_0 t}{4}}\left[\frac{J_{\bar{\nu}}\left(\xi e^{\frac{-b_0 t}{2}}\right)}{J_{\bar{\nu}}\left(\xi e^{\frac{-b_0 \bar{t}}{2}}\right)} - \frac{Y_{\bar{\nu}}\left(\xi e^{\frac{-b_0 t}{2}}\right)}{Y_{\bar{\nu}}\left(\xi e^{\frac{-b_0 \bar{t}}{2}}\right)}\right], \tag{33}$$

where $\bar{\nu} = -1/2\sqrt{\frac{b_0^2 - 16w^2}{b_0^2}}$ and $\xi = \frac{2\lambda_i \Omega^{1/2}}{b_0}$.

With the solutions anyone can calculate the Bogoliubov coefficient that supplies the vacuum excitation using $(\phi_{in}, \phi^*_{out})$ and imposing some appropriate boundary conditions in order to find the excitation modes.

We emphasize that in this context $b(t)$ is not understood as a *radion* field [11], since it has not a continuum behavior at different times in our model. Again, our interest here is restricted to some characteristics of a combined effect of a strange kinematics with a unusual topology.

V. CONCLUDING REMARKS AND OUTLOOKS

We have investigated how a topology where one or more dimensions are compactified can present signatures in some physical aspects, like vacuum fluctuation and particle production. It can be realized that the topology generated by the cosmic string is codified in the deficit angle and in the cylindrical symmetry, while the S^1 compactification is realized in all components of stress tensor, and generates a new effect (new component) as well as in the transformation from continuous modes in discrete ones. A general analysis of the teleparallel geometry corrected Klein-Gordon massless field is done, and we also investigated the generalization of extra-dimensional toroidal-compactified models. One interesting particular case, when the topology is given by $\mathbb{R}^3 \times T^2$, can introduce a Kaluza-Klein theory over AdS spacetime, since $\mathbb{R}^3 \times T^2 \simeq (\mathbb{R}^3 \times S^1) \times S^1$, and $\mathbb{R}^3 \times S^1$ is the AdS topology. Then, in this topology AdS $\times S^1$, it can be also possible to use all the results arising from Eq.(28). In this case obviously the Kaluza-Klein modes are discrete, since we investigate compactified extra dimensions. The cases presented in Sections (II-IV) give rise to the most direct generalization of our previous analysis and play an important role in this context, since have interesting properties in the particle production processes.

Since Eqs.(5) and (8) do not equal each other, the Lorentz invariance is not preserved in the z direction at the quantum level, an effect caused by the nontrivial topology. Also, by accomplishing a toroidal compactification, there naturally arises a maximal number of covariantly constant spinors, since flat tori are the only manifolds with trivial holonomy. Each one of these spinors is related to a supersymmetry that remains unbroken by the compactification. Supersymmetric cosmic strings models involving toroidal compactification are beyond the scope of this paper.

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